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BUCKLING OF LONG, REGULAR TRUSSES

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Abstract—The buckling of pin-jointed trusses due to axial end loading is examined using finite difference calculus. Exact solutions are found from which approximate formulae are derived which may be compared with the critical loads for analogous columns buckling due to bending and shear. These are found to differ from previously proposed analogies which were established on an intuitive basis. The possibility of analogies for the lateral buckling of trusses is also examined.

1. INTRODUCTION

ANALOGIES have previously been drawn [1] between the behaviour of pin-jointed trusses in compression and the behaviour of columns buckling due to bending and shear. The author has previously shown [2] that the deflexions of laterally loaded trusses are closely related to the bending and shear deflexions of beams. The method used avoids the common intuitive approaches but relies instead on the rigorous derivation of a differential equation for the truss from exact finite difference equations for a typical truss module. In this paper, these finite difference equations will again be formed, allowing for large axial forces in the bars. Taking skew-symmetrical end conditions, sinusoidal deflexion modes can be found, leading to expressions for the critical axial loading. If the wavelength is taken as large in comparison with the module size, then these expressions reduce to forms which may be compared with the expression for the critical load of a column, buckling due to bending and shear. These are similar but not identical to the previous intuitive expressions [1].

An examination is also made of the lateral buckling of rigid-jointed plane trusses. This is done by taking the axial force in any member to be small in comparison with its Euler buckling load, since overall buckling of the truss is sought rather than the local buckling of individual members. Livesley's [3] stability functions can then be expressed by truncated expansions in terms of the axial load. Assuming large, equal and opposite axial forces in the upper and lower chords of the truss and skew symmetrical end conditions, a buckling mode can again be found. This leads to a comparison between the resulting expression for the buckling load with that for the lateral buckling of beams in pure bending.

2. THE MEMBER STIFFNESS EQUATIONS

The relationship between the end loads and deflexions of a member arising from incipient buckling will now be considered. The initial compression force P in the member will be large in comparison with these end loads and the second order effects which this force produces will be included.

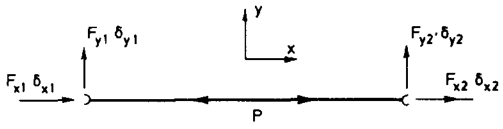

FIG. 1

If the member of a plane frame is pin-ended, Fig. 1, and has a length l Young's modulus E and cross-sectional area A, the relationships between the in-plane end forces F_{x1}, F_{x2}, F_{y1} and F_{y2} and the corresponding end displacements δ_{x1}, δ_{x2}, δ_{y1} and δ_{y2} are

$$\begin{bmatrix} F_{x1} \\ F_{y1} \end{bmatrix} = \begin{bmatrix} \frac{EA}{l} & 0 \\ 0 & -\frac{P}{l} \end{bmatrix} \begin{bmatrix} \delta_{x1} \\ \delta_{y1} \end{bmatrix} + \begin{bmatrix} -\frac{EA}{l} & 0 \\ 0 & \frac{P}{l} \end{bmatrix} \begin{bmatrix} \delta_{x2} \\ \delta_{y2} \end{bmatrix}, \tag{1}$$

$$\begin{bmatrix} F_{x2} \\ F_{y2} \end{bmatrix} = \begin{bmatrix} -\frac{EA}{l} & 0 \\ 0 & \frac{P}{l} \end{bmatrix} \begin{bmatrix} \delta_{x1} \\ \delta_{y1} \end{bmatrix} + \begin{bmatrix} \frac{EA}{l} & 0 \\ 0 & -\frac{P}{l} \end{bmatrix} \begin{bmatrix} \delta_{x2} \\ \delta_{y2} \end{bmatrix}. \tag{2}$$

If the member co-ordinates x, y are at an anticlockwise angle α to an overall set of co-ordinates x^*, y^* the above matrices can be related to the overall system using the transformations

$$\begin{bmatrix} F^*_{xi} \\ F^*_{yi} \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} F_{xi} \\ F_{yi} \end{bmatrix}, \quad \begin{bmatrix} \delta_{xi} \\ \delta_{yi} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \delta^*_{xi} \\ \delta^*_{yi} \end{bmatrix}, \tag{3}$$

where F^*_{xi}, F^*_{yi}, δ^*_{xi} and δ^*_{yi} ($i = 1$ or 2) are the end forces and displacements in the directions of the overall co-ordinates.

If the member is part of a rigid-jointed frame, then its flexural and torsional behaviour must be taken into account. The relevant flexural stiffness of a member will be denoted by EI and its torsional stiffness by GJ. In order to obtain stiffness matrices which are linear functions of the axial compression P, Livesley's stability functions will be expanded in powers of P, taking only the first two terms. Then

$$\begin{aligned} \phi_2 &= 1 - \frac{1}{60}\frac{Pl^2}{EI}; \qquad \phi_3 = 1 - \frac{1}{30}\frac{Pl^2}{EI}; \\ \phi_4 &= 1 + \frac{1}{60}\frac{Pl^2}{EI}; \qquad \phi_5 = 1 - \frac{1}{10}\frac{Pl^2}{EI}. \end{aligned} \tag{4}$$

The maximum error involved in using these simplified expressions in the range of P between ± 40 per cent of the pin-ended Euler buckling load (or ± 10 per cent of the fixed-ended buckling load) is 0·9 per cent. If, for example, they are used to determine the lowest in-plane buckling load of a rectangular portal frame with all members equal and equal axial forces in the stanchions, the critical axial forces are found to be 7·01 EI/l^2. This compares well with the exact answer of 7·38 EI/l^2.

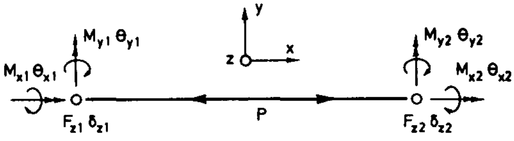

FIG. 2.

In this article, only the out-of-plane buckling of plane rigid-jointed frames will be considered. The plane of the frame will again be taken as the xy plane shown in Fig. 2. The end moment induced by buckling are M_{x1}, M_{y1}, M_{x2} and M_{y2} and the corresponding rotations are θ_{x1}, θ_{y1}, θ_{x2} and θ_{y2}. End forces F_{z1} and F_{z2} also arise and the corresponding end displacements are δ_{z1} and δ_{z2}. Taking the y and z planes to be planes of symmetry of the member's cross-section, and ignoring the small effect of axial force on torsional stiffness, the stiffness relationships between the above end loads and deflexions are

$$\begin{bmatrix} M_{x1} \\ M_{y1} \\ F_{z1} \end{bmatrix} = \begin{bmatrix} l\gamma & 0 & 0 \\ 0 & 4l(\beta-2p) & -6(\beta-p) \\ 0 & -6(\beta-p) & 12(\beta-6p)/l \end{bmatrix} \begin{bmatrix} \theta_{x1} \\ \theta_{y1} \\ \delta_{z1} \end{bmatrix} + \begin{bmatrix} -l\gamma & 0 & 0 \\ 0 & 2l(\beta+p) & 6(\beta-p) \\ 0 & -6(\beta-p) & -12(\beta-6p)/l \end{bmatrix} \begin{bmatrix} \theta_{x2} \\ \theta_{y2} \\ \delta_{z2} \end{bmatrix}, \tag{5}$$

$$\begin{bmatrix} M_{x2} \\ M_{y2} \\ F_{z2} \end{bmatrix} = \begin{bmatrix} -l\gamma & 0 & 0 \\ 0 & 2l(\beta+p) & -6(\beta-p) \\ 0 & 6(\beta-p) & -12(\beta-6p)/l \end{bmatrix} \begin{bmatrix} \theta_{x1} \\ \theta_{y1} \\ \delta_{z1} \end{bmatrix} + \begin{bmatrix} l\gamma & 0 & 0 \\ 0 & 4l(\beta-2p) & 6(\beta-p) \\ 0 & 6(\beta-p) & 12(\beta-6p)/l \end{bmatrix} \begin{bmatrix} \theta_{x2} \\ \theta_{y2} \\ \delta_{z2} \end{bmatrix}, \tag{6}$$

where

$$\beta = EI/l^2; \qquad \gamma = GJ/l^2; \qquad p = P/60. \tag{7}$$

If the member co-ordinates x, y are at an angle α to an overall set of co-ordinates x^*, y^* the transformation matrices in this case are

$$\begin{bmatrix} M^*_{xi} \\ M^*_{yi} \\ F^*_{zi} \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M_{xi} \\ M_{yi} \\ F_{zi} \end{bmatrix}, \quad \begin{bmatrix} \theta_{xi} \\ \theta_{yi} \\ \delta_{zi} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta^*_{xi} \\ \theta^*_{yi} \\ \delta^*_{zi} \end{bmatrix}, \tag{8}$$

where the end forces and displacements in terms of the overall co-ordinate system are marked with an asterisk and i is 1 or 2.

Stiffness equations for either type of truss can now be formed in the usual manner, using the conditions of joint equilibrium and compatability of joint deflexions.

3. THE FINITE DIFFERENCE OPERATORS

The upper and lower joints will each be numbered from left to right and the joint deflexions taken as functions of the joint number. The operator E will be taken to produce a step to the right in the joint number. Thus if X is some general joint number and $f(X)$ a function of it, then

$$Ef(X) = f(X+1). \tag{9}$$

More generally,

$$E^N f(X) = f(X+N), \tag{10}$$

where N in the present paper may be a positive or negative integer or fraction. The symbol $\boxtimes$ will also be used where

$$\boxtimes = 2-E-E^{-1}. \tag{11}$$

These operators can be expanded using Taylor's series so that if a unit step in X corresponds to an increase of λ in x,

$$E^N = 1+N\lambda\frac{\mathrm{d}}{\mathrm{d}x}+\frac{(N\lambda)^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\dots+\frac{(N\lambda)^n}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}, \tag{12}$$

$$\boxtimes = -2\left[\frac{\lambda^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{\lambda^4}{4!}\frac{\mathrm{d}^4}{\mathrm{d}x^2}\dots+\frac{\lambda^{2n}}{2n!}\frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}\right]. \tag{13}$$

Sinusoidal buckling modes will be examined so that the effect of these operators on sinusoidal functions needs to be known. For example

$$E^N \sin(kX+\alpha) = \sin(kX+kN+\alpha), \tag{14}$$

$$\begin{aligned}\boxtimes \sin(kX+\alpha) &= 2\sin(kX+\alpha)-\sin(kX+k+\alpha)-\sin(kX-k+\alpha)\\ &= 2(1-\cos k)\sin(kX+\alpha).\end{aligned} \tag{15}$$

4. IN-PLANE BUCKLING OF PRATT AND HOWE TRUSSES

Figure 3 shows part of a regular, pin-jointed Pratt or Howe truss. The axial stiffness (EA/l) for the top and bottom bars will be taken as equal to t, those of the diagonal bars as equal to d and those of the vertical bars as equal to v. Further quantities will now be

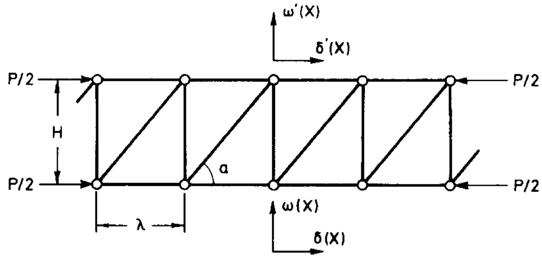

FIG. 3.

defined in terms of these and the angle α of the diagonal bars:

$$e = d\cos^2\alpha; \qquad r = \tan\alpha; \qquad \tau = t/e; \qquad \nu = \upsilon/er^2; \qquad \rho = P/(2er^2\lambda). \tag{16}$$

Taking half the total axial force P to be acting in the top bars and half to be acting in the lower bars, the stiffness equations for the equilibrium of the upper and lower joints, when no external forces act on these joints, can be written as

$$\begin{bmatrix} 1+\tau\boxbslash & 1 & -1 & -1 \\ 1 & \nu+1-\rho\boxbslash & -1 & -E\nu-1 \\ -1 & -1 & 1+\tau\boxbslash & 1 \\ -1 & -E^{-1}\nu-1 & 1 & \nu+1-\rho\boxbslash \end{bmatrix} \begin{bmatrix} E\delta' \\ rE\omega' \\ \delta \\ r\omega \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{17}$$

where δ' and ω' are the horizontal and vertical displacements of a typical lower joint, δ and ω are the corresponding displacements of the upper joint. This equation may be compared with equation (8) of Ref. [2].

For joint deflexions to occur without external joint loads, (apart from the axial compression), the determinant of the above matrix operating on a representative displacement function $\phi(X)$ must be zero, giving

$$\tau\boxbslash^2\{-4\rho\nu+\boxbslash[\tau\nu-2\rho\tau(\nu+1)+2\rho^2]+\boxbslash^2\rho^2\tau\}\phi(X) = 0. \tag{18}$$

Equation (18) is the necessary condition for a nonzero solution to exist. Such an equation has been used previously (7) to solve the torsional–flexural buckling equations.

The condition

$$\boxbslash^2\phi(X) = 0, \tag{19}$$

is satisfied by

$$\phi = A+BX+CX^2+DX^3. \tag{20}$$

The particular solutions corresponding to the case when C and D are both zero correspond to rigid-body movements of the truss and axial compression. These can be written in the form

$$\begin{aligned} E\delta' &= A_1+B_1X; & rE\omega' &= A_2+B_2X; \\ \delta &= A_3+B_3X; & r\omega &= A_4+B_4X. \end{aligned} \tag{21}$$

The coefficients A_i and B_i are to some extent related by equation (17). For a rigid-body movement U in the direction of δ and δ', V in the direction of ω and ω' and on anticlockwise rotation Θ about a point on the middle line of the truss at the origin,

$$\begin{aligned} A_1 &= U-H\Theta/2; & A_3 &= U+H\Theta/2; & B_1 &= B_3 = 0; \\ A_2 &= rV+H\Theta; & A_4 &= rV; & B_2 &= B_4 = H\Theta. \end{aligned} \tag{22}$$

The complete solution of equation (20) also allows an internal end moment and shear force to be applied, also without external loads on the intermediate joints.

The remaining solutions of equation (18) result from the operation of the terms inside the curly brackets on $\phi(X)$ giving zero. Suppose that $\phi(X)$ has a general sinusoidal form given by $A\sin(kX+a)$. Then from equations (15) and (18), $\phi(X)$ is nonzero if

$$P_c = \frac{\lambda}{c(ct+e)}\{[ve+ct(v+er^2)] \pm \{[ve+ct(v+er^2)]^2 - 2c^2er^2tv(ct+e)\}^{\frac{1}{2}}\}, \tag{23}$$

where P_c is the critical load at which this waveform becomes possible and c is $1-\cos k$. The deflexions $E\delta'$, $rE\omega'$, δ and $r\omega$ can be expressed in terms of functions of the form $A_i\sin(kX+a_i)$, where i takes the integer values 1–4. Relationships between the coefficients can be established by substituting these functions into equation (17). A particular solution of interest here is given by

$$\delta = A\cos\frac{\pi\lambda}{L}(X+\tfrac{1}{2}), \tag{24}$$

$$\delta' = -A\cos\frac{\pi\lambda}{L}(X-\tfrac{1}{2}), \tag{25}$$

$$\omega_m = \tfrac{1}{2}(\omega+\omega') = \frac{A(2+\tau c)}{\rho cr}\sin\frac{\pi\lambda}{2L}\sin\frac{\pi\lambda}{L}X, \tag{24}$$

$$\tfrac{1}{2}(\omega-\omega') = \frac{-A(1+\tau c)}{r}\cos\frac{\pi\lambda}{L}(X+\tfrac{1}{2}), \tag{25}$$

where ω_m is the displacement of the middle line of the truss and k takes the value $\pi\lambda/L$. An expression for the bending moment M produced about the middle line can be obtained from the above results and is given by

$$M = 2AHt\sin\frac{\pi\lambda}{2L}\sin\frac{\pi\lambda}{L}X, \tag{26}$$

taking a hogging moment as positive. This solution gives the displacement of the middle line of the truss and the bending moment about the middle line as zero at x equal to zero and L. These conditions correspond to analogous pin-ended conditions and the above solution is possible provided that equation (23) is satisfied for the value of c corresponding to k equals $\pi\lambda/L$.

Instead of using the exact result given by equation (23), a simplifying approximation can be made which produces a form comparable with earlier results. Since $L \gg \lambda$, k will be small so that approximately

$$c = \frac{\pi^2\lambda^2}{2L^2}\left(1-\frac{\pi^2\lambda^2}{12L^2}\right), \tag{27}$$

using the first three terms in the expansion of $\cos k$. Equation (23) can be expanded by means of the binomial theorem using the condition that c is small, leading to an expression for the lowest critical load

$$P_c \doteqdot \frac{\pi^2 EI^*}{L^2}\left[1+\frac{\pi^2 EI^*}{L^2}\left(\frac{r^2}{v}+\frac{1}{e}+\left\{\frac{1}{6t}\right\}\right)\frac{1}{\lambda r^2}\right], \tag{28}$$

where

$$EI^* = t\lambda H^2/2, \tag{29}$$

is the bending stiffness of an imaginary beam formed by the two chords. This result agrees with that obtained by Engesser [4] apart from the last term, shown in curly brackets, which results from the second term in the expansion for c given by equation (27)

5. IN-PLANE BUCKLING OF CROSS-BRACED TRUSSES

A typical cross-braced truss is shown in Fig. 4. Since it is redundant, the proportion of the total axial force carried by the chord members (top and bottom bars) is affected. For this reason, the vertical bars cannot be ignored as suggested by Timoshenko [1].

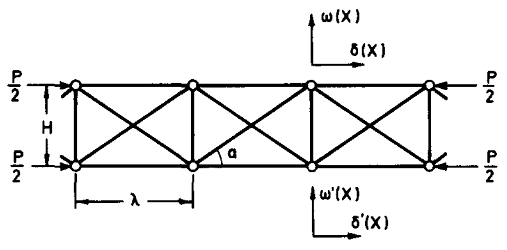

FIG. 4.

The notation used in the previous section, equation (16) will again be used except that instead of ρ, the compressive forces in the chords, diagonals and vertical bars will be expressed by $f\lambda r^2$, $g\lambda \sec^3 \alpha$ and $-h\lambda r$ respectively. These can be found in terms of the total axial force P by strain energy methods for example, giving

$$f = \left[\frac{t}{er^2}+\frac{t}{v}\right]h;$$

$$g = h/(1+r^2);$$

$$h = P/2\lambda\left(1+\frac{tr^2}{v}+\frac{t}{e}\right). \tag{30}$$

The stiffness matrix equation is found by the same method used in the previous section and equating its determinant operating on ϕ to zero gives

$$\begin{vmatrix} t\boxslash+h+2(e-gr^2) & 0 & -h-(e-gr^2)(E+E^{-1}) & (e+g)(E^{-1}-E) \\ 0 & -f\boxslash+\dfrac{v}{r^2}+2(e-g/r^2) & (e+g)(E^{-1}-E) & -\dfrac{v}{r^2}-(e-g/r^2)(E+E^{-1}) \\ -h-(e-gr^2)(E+E^{-1}) & (e+g)(E-E^{-1}) & t\boxslash+h+2(e-gr^2) & 0 \\ (e+g)(E-E^{-1}) & -\dfrac{v}{r^2}-(e-g/r^2)(E+E^{-1}) & 0 & -f\boxslash+\dfrac{v}{r^2}+2(e-g/r^2) \end{vmatrix}\phi = 0. \quad (31)$$

Equations (14) and (15) can again be used, assuming a sinusoidal buckling mode, and the above determinant reduces to a product of two quadratic expressions in terms of the axial force. The lowest critical load predicted by these expressions is

$$P_c = \lambda r^2[B-\sqrt{(B^2-4AC)}]/sA \doteqdot 2\lambda r^2 C/s\left(B-\frac{AC}{B}\right), \quad (32)$$

where

$$A = [r^2-1-cr^2+s(c-1)]/(1+r^2);$$

$$B = e(2-c)+se(1-c)+ct\left(1-\frac{s}{1+r^2}\right); \quad (33)$$

$$C = ects;$$

$$s = ver^2/[ve+t(v+er^2)].$$

The approximation in equation (32) is possible when $L \gg \lambda$ so that C is much smaller than A and B, permitting the use of a short binomial expansion. Using this result in conjunction with equation (27) then gives

$$P_c \doteqdot \frac{\pi^2 EI^*/L^2}{1+\dfrac{s}{2}+\dfrac{\pi^2 EI^*}{L^2}\left[\dfrac{1}{2+s}-\dfrac{e}{12t}(4+5s)\right]\dfrac{1}{\lambda er^2}} \quad (34)$$

where EI^* is again defined by equation (29). Taking the particular case when the stiffness of the battens, v, is zero the above equation reduces to

$$P_c \doteqdot \frac{\pi^2 EI^*/L^2}{1+\dfrac{\pi^2 EI^*}{L^2}\left[\dfrac{1}{2e}-\left\{\dfrac{1}{3t}\right\}\right]\dfrac{1}{\lambda r^2}}. \quad (35)$$

This corresponds to the result given by Timoshenko [1] who assumes that the effect of the battens can be neglected. It agrees with his result apart from the term in curly brackets which results only in part from the second term in the expansion for c (equation 27).

6. IN-PLANE BUCKLING OF WARREN TRUSSES

Figure 5 shows part of a typical Warren truss axially loaded by force P. Being statically determinate, the analysis of this truss follows the same lines as that given in Section 4. Using the notation given by equation (16), the stiffness matrix equation can be written as

$$\begin{bmatrix} 2+\tau\boxbslash & 0 & -(E^{\frac{1}{2}}+E^{-\frac{1}{2}}) & (E^{-\frac{1}{2}}-E^{\frac{1}{2}}) \\ 0 & 2-\rho\boxbslash & (E^{-\frac{1}{2}}-E^{\frac{1}{2}}) & -(E^{\frac{1}{2}}+E^{-\frac{1}{2}}) \\ -(E^{\frac{1}{2}}+E^{-\frac{1}{2}}) & (E^{\frac{1}{2}}-E^{-\frac{1}{2}}) & 2+\tau\boxbslash & 0 \\ (E^{\frac{1}{2}}-E^{-\frac{1}{2}}) & -(E^{\frac{1}{2}}+E^{-\frac{1}{2}}) & 0 & 2-\rho\boxbslash \end{bmatrix} \begin{bmatrix} \delta' \\ \omega' r \\ \delta \\ \omega r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{36}$$

As before, the buckling load is found from the condition that the determinant of this matrix, operating on a sinusoidal displacement function, is zero.

The exact solution for this case is particularly simple and the lowest critical load is given by

$$P_c = \frac{2er^2\lambda[1-\cos(\pi\lambda/2L)]}{1-\cos(\pi\lambda/L)+e/t[1+\cos(\pi\lambda/2L)]}. \tag{37}$$

In the case when the diagonals are infinitely stiff, this result becomes

$$P_c = \frac{\pi^2 EI^*}{L^2}\left(\frac{4L}{\pi\lambda}\tan\frac{\pi\lambda}{4L}\right)^2, \tag{38}$$

which coincides exactly with the result obtained by von Mises and Ratzersdorfer [5] for this case.

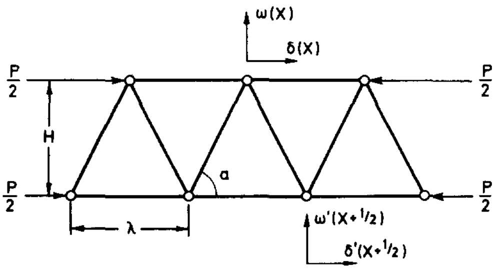

FIG. 5.

Using the methods of approximation outlined earlier, equation (37) becomes

$$P_c = \frac{\pi^2 EI^*}{L^2}\left[1+\frac{\pi^2 EI^*}{L^2}\left(\frac{2}{e}-\left\{\frac{1}{3t}\right\}\right)\frac{1}{\lambda r^2}\right]. \tag{39}$$

This again corresponds with the expression proposed by Timoshenko [1] except for the term in curly brackets. As in Section 5, this term results only in part from the expansion given by equation (27).

7. OUT-OF-PLANE BUCKLING OF A WARREN TRUSS

The question of whether the lateral buckling of trusses is in any way analogous to that of beams will now be examined. Figure 6 shows a rigid-jointed Warren truss loaded by equal and opposite forces P in its two chords. These can be taken as the primary forces

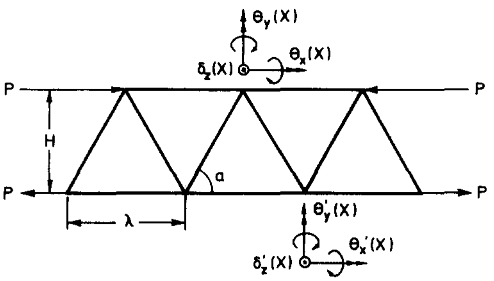

FIG. 6.

resulting from a large, uniform bending moment M equal to PH applied to the truss. The form of the matrix equations for such problems was discussed in Section 2 and the resulting stiffness matrix equation for a typical unit of the truss is given by

$$\begin{bmatrix} a & 0 & b & j & k & -l \\ 0 & c & d & k & m & -n \\ b & -d & e & l & n & q \\ \bar{j} & \bar{k} & \bar{l} & a & 0 & -b \\ \bar{k} & \bar{m} & \bar{n} & 0 & c^* & d^* \\ -\bar{l} & -\bar{n} & \bar{q} & -b & -d^* & e^* \end{bmatrix} \begin{bmatrix} \theta'_x \\ \theta'_y \\ \delta'_z/\lambda \\ \theta_x \\ \theta_y \\ \delta_z/\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{40}$$

where

$$\begin{aligned} a &= \gamma_c \boxbslash + 4\beta_d \sin\alpha\tan\alpha + \gamma_d \cos\alpha;\ b = 12\beta_d \sin\alpha; \\ c &= 2\beta_c(6-\boxbslash) + 2p(6+\boxbslash) + 4\beta_d\cos\alpha + \gamma_d \sin\alpha\tan\alpha; \\ d &= 6(\beta_c + p)(E - E^{-1});\ e = 12\boxbslash(\beta_c + 6p) + 48\beta_d\cos\alpha; \\ j &= (\beta_d \sin\alpha\tan\alpha - \tfrac{1}{2}\gamma_d\cos\alpha)(E^{-1}+1); \\ k &= \sin\alpha(\beta_d + \tfrac{1}{2}\gamma_d)(E^{-1}-1);\ l = 6\beta_d\sin\alpha(E^{-1}+1); \\ m &= (\beta_d\cos\alpha - \tfrac{1}{2}\gamma_d \sin\alpha\tan\alpha)(E^{-1}+1); \\ n &= 6\beta_d\cos\alpha(E^{-1}-1);\ q = -24\beta_d\cos\alpha(E^{-1}+1), \end{aligned} \tag{41}$$

the notation of equation (7) being used where the subscripts c and d refer to the chord and diagonal members respectively, a barred term indicates that E^{-1} is replaced by E and a starred term indicates that p is replaced by $-p$.

A possible buckling mode for such a truss is again given by taking sinusoidal expressions of equal wavelength for each of the deflexions. These expressions will be physically in phase except those for θ'_y and θ_y which will be $\pi/2$ out of phase with the rest. Again, such a mode can be chosen to fit skew-symmetrical boundary conditions L apart, provided for example by vertical roller bearings, and leads to the lowest value of the critical moment when L is half a wavelength.

The general result is extremely complex and only a particular case will be considered here. All the members will be taken to have equal lengths and bending stiffness EI and their torsional stiffnesses assumed to be negligible. The exact expression for the critical moment is then given by

$$M_c = \frac{\sqrt{3}}{2}P_c\lambda = \sqrt{3}\frac{EI}{\lambda}\{[B-(B^2-1800AC)^{\frac{1}{2}}]/A\}^{\frac{1}{2}}, \tag{42}$$

where

$$\begin{aligned} A &= 24 + 28c + 10c^2 + c^3; \\ B &= 14625 - 2286c - 907c^2 - 28c^3; \\ C &= 30c + 17c^2 + 2c^3; \\ c &= 1 - \cos\pi\lambda/L. \end{aligned} \tag{43}$$

Again using the expansion for c given by equation (27) and the binomial theorem, the above result can be written as

$$M_c \doteqdot \frac{\pi}{L}(1{\cdot}664EI)\left[1 - 9{\cdot}127\frac{\pi^2\lambda^2}{L^2}\right]^{\frac{1}{2}}. \tag{44}$$

However, the expression for the critical moment of a beam with lateral bending stiffness EI^*, torsional stiffness GJ^*, nonuniform torsional stiffness EK^* and length L is

$$M_c = \frac{\pi}{L}(EI^*GJ^*)^{\frac{1}{2}}\left[1 + \frac{EK^*}{GJ^*}\frac{\pi^2}{L^2}\right]^{\frac{1}{2}}. \tag{45}$$

Comparing equations (44 and 45) term by term would require the postulation of a negative equivalent stiffness. This turns out to be unhelpful, since it leads to imaginary terms in the expressions for other lateral buckling loads. At best then, it is possible to compare the first order results given by ignoring the last term in each equation, leading to a value for an equivalent EI^*GJ^*. If this equivalence holds good for the problem of the lateral buckling of a cantilever subject to an end shear force P_c at the tip, the result for the truss should be

$$P_c \doteqdot 6{\cdot}68EI/L^2. \tag{46}$$

Computer results for such cantilevered Warren trusses have been found (6) and fit very closely to straight lines on graphs of $\log P_c$ plotted against number of bays in the cantilever. For example, the line fitting the results for an end load P_c on the upper chord of such a truss is

$$P_c \doteqdot 7{\cdot}85EI/L^2. \tag{47}$$

As might be expected from energy considerations, the values of P_c are higher when it acts at the end of the lower chord, so that for the computer results found, equation (46) gives a safe estimate of P_c.

8. CONCLUDING REMARKS

A generally applicable method of finding approximations to the buckling loads of long trusses has been given. Slight corrections to earlier results for the in-plane buckling of pin-jointed trusses have been found, but the first order (bending) terms shown to be usually correct. An analogy between the lateral buckling of beams and trusses has been investigated and it was shown that, at best, only a first order analogy could be established. Exact results for the above problems were also given although, as a rule, these proved to be more difficult to apply.

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Резюме—Исчислением конечной разницы исследуется коробление ферм с шпилечными соединениями вследствие нагрузки на осевой конец. Получили точные решения на которых построили приближенные формулы, которые можно сравнить с моделями колонн, покоробленных вследствие изгиба и сдвига. Нашли, что они отличаются от прежде предложенных аналогий, которые были установлены на интуитивном основании. Также рассматривается возможность поперечного изгиба ферм.